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A SURVEY OF ONE-STEP SPLITTING METHODS FOR SEMI-DISCRETE FIRST ORDER HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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A survey of one-step splitting methods for semi-discrete first order hyperbolic partial differential equations *)

by

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ABSTRACT

In this report splitting methods are discussed for first order hyperbolic partial differential equations via the method of lines, and in particular the time integration will be discussed. A class of one-step integration formulas is defined, which is shown to contain several well-known splitting methods. For a number of methods stability results are given.

KEY WORDS & PHRASES: *Numerical analysis; ordinary differential equations; partial differential equations; method of lines; splitting methods*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

It is the purpose of this report to discuss splitting methods for first order multi-dimensional hyperbolic partial differential equations (PDE's) via the *method of lines* (cf. [16]), and in particular the time-integration. Well-known splitting methods are the alternating direction implicit methods (ADI) [9,10], the locally -one- dimensional methods (LOD) [10,18] and the hopscotch methods [8,12], which are usually formulated and analysed as direct grid methods.

Let

$$(1.1) \qquad \frac{dy}{dt} = f(t,y), \quad f: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

denote a semi-discrete first order hyperbolic PDE, where we assume that f can be linearly split into k terms, k > 1 i.e.,

$$(1.2) \qquad f(t,y) = \sum_{i=1}^{k} f_i(t,y), \quad f_i: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N.$$

The functions f_i are called *splitting functions* [16] and depend on the original PDE and the type of semi-discretization.

In mathematical physics (cf. [20]) hyperbolic initial-boundary value problems occur frequently, for instance, in oceanography, meteorology, seismology, aerodynamics and fluid dynamics. The general treatment of (nonlinear) hyperbolic equations is complicated by the great variety of physical phenomena that are important in different situations. This means that it is usually necessary to construct special methods which are suitable for the particular problem concerned. Even with smooth problems one must be careful of non-linear instabilities [22]. Limitations of the methods relate to the dispersion and dissipation present in the numerical approximation [23]. The implicit methods are mostly nondissipative (cf. [22, p.109]) and hence of dubious value for nonlinear hyperbolic systems in which discontinuities occur. In fluid dynamics the system can change significantly in a time Δt (shocks), so that there is no motivation for using long time intervals. Implicit methods usually have larger ranges of stability than explicit methods.

It is not always clear what boundary conditions must be imposed to

ensure a well-posed problem, which is more severe in multi-dimensional problems than in one-dimensional problems [2,19,20,22]. Implicit methods need often more boundary data than is necessary for the problem to be well-posed. For one- and multi-dimensional hyperbolic equations a discussion of stable approximations for the boundary conditions can be found in [2,14,20].

In section 2 of this report, we define a general class of one-step integration formulas for the systems (1.1)-(1.2), which we shall call *splitting formulas*. We distinguish between splitting functions and splitting formulas, and a combination of both will be called a *splitting method* [16]. Several examples of known linear splitting methods, considered in this way, are discussed in the sections 3 and 4.

In section 5 the stability properties of the one-step splitting methods, discussed in the sections 3 and 4, are investigated by making use of matrix theory. The results are presented in a uniform way and are based on a basic theorem.

2. LINEAR SPLITTING FORMULAS

Consider the m-stage, one-step integration formula (cf. [16])

$$
\begin{aligned}
y_{n+1}^{(0)} &= y_n, \\
y_{n+1}^{(j)} &= y_n + \tau_n \sum_{\ell=0}^{j} \sum_{i=1}^{k} \lambda_{j\ell i} f_i(t_n + \alpha_{j\ell i}\tau_n, y_{n+1}^{(\ell)}), \quad j = 1(1)m, \\
y_{n+1} &= y_{n+1}^{(m)},
\end{aligned}
\tag{2.1}
$$

where y_n denotes the numerical approximation at $t = t_n$ and $\tau_n = t_{n+1} - t_n$. Each formula, belonging to (2.1), is called a *linear splitting formula*. The parameters $\lambda_{j\ell i}$ and $\alpha_{j\ell i}$ serve to make this scheme a consistent and stable approximation to (1.1). In particular, however, they should be used to exploit the splitting property (1.2) in order to obtain an attractive computational process.

REMARK 2.1. Observe that (2.1) is closely related to Runge-Kutta methods and if $\lambda_{\ldots jji} = 0$, the resulting scheme is explicit. Examples of *explicit* splitting methods for hyperbolic PDE's can be found in [6,25]. Such methods are related to explicit Runge-Kutta methods for ordinary differential equations (ODE's). The Strang splitting schemes [25,26] for multi-dimensional hyperbolic equations are based on the idea of representing multi-dimensional difference operators as a product of one-dimensional operators, especially the one-dimensional Lax-Wendroff operator is chosen (cf. [22, p.302]). The Lax-Wendroff schemes can not be obtained via the method of lines approach because they involve manipulation of terms in both the time and space discretizations.

The order conditions of (2.1) can be derived through a formal Taylor expansion and conditions up to order 3 are listed in table 2.1.

For convergence of a p-th order consistent scheme (2.1) we refer to [16], where some references are given concerning convergence results for one-step formulas defined by general increment functions.

Table 2.1. Consistency conditions for (2.1).

p=1	$\sum_{\ell=0}^{m} \lambda_{m\ell i} = 1,$	$i = 1(1)k,$
p=2	$\sum_{\ell=1}^{m} \sum_{r=0}^{\ell} \lambda_{m\ell i}\lambda_{\ell rj} = \frac{1}{2},$	$i,j = 1(1)k,$
	$\sum_{\ell=0}^{m} \lambda_{m\ell i}\alpha_{m\ell i} = \frac{1}{2},$	$i = 1(1)k,$
p=3	$\sum_{r=0}^{\ell} \lambda_{\ell rj} = C_{\ell},$	$j = 1(1)k,\ \ell = 1(1)m,$
	$\sum_{\ell=1}^{m} \lambda_{m\ell i}(C_{\ell})^2 = \frac{1}{3},$	$i = 1(1)k,$
	$\sum_{\ell=1}^{m} \sum_{r=1}^{\ell} \sum_{s=0}^{r} \lambda_{m\ell i}\lambda_{\ell rj}\lambda_{rsg} = \frac{1}{6},$	$i,j,g = 1(1)k,$
	$\sum_{\ell=0}^{m} \lambda_{m\ell i}(\alpha_{m\ell i})^2 = \frac{1}{3},$	$i = 1(1)k,$
	$\sum_{\ell=1}^{m} \sum_{r=0}^{\ell} \lambda_{m\ell i}\alpha_{m\ell i}\lambda_{\ell rj} = \frac{1}{3},$	$i,j = 1(1)k,$
	$\sum_{\ell=1}^{m} \sum_{r=0}^{\ell} \lambda_{m\ell i}\alpha_{\ell rj}\lambda_{\ell rj} = \frac{1}{6},$	$i,j = 1(1)k.$

3. TWO-TERM SPLITTING METHODS

In this section we give a survey of important linear splitting methods for two-dimensional hyperbolic equations, which are generally formulated and analysed as so-called *direct grid methods*.

Let us consider two-dimensional hyperbolic equations of the form

$$u_t = Au_x + Bu_y, \tag{3.1a}$$

$$u_t = A(x,y)u_x + B(x,y)u_y, \tag{3.1b}$$

$$u_t = A(u)u_x + B(u)u_y + C(x,y)u, \tag{3.1c}$$

$$u_t = A(x,y,t)u_x + B(x,y,t)u_y, \tag{3.1d}$$

$$u_t = \frac{\partial}{\partial x} G_1(u) + \frac{\partial}{\partial y} G_2(u). \tag{3.1e}$$

In general, the unknown u may be either a scalar or vector function of x,y,t and A and B may be scalars or matrices. The solution of (3.1a)-(3.1e) is required in an arbitrary region $\Omega \times (0,T]$ with suitable boundary conditions on $\delta\Omega \times (0,T]$, were Ω is normally a bounded and path-connected region in the two-dimensional (x,y)-space with sides parallel to the coordinate axes, $\delta\Omega$ is the boundary of Ω, and (0,T] is the time interval $0 < t \leq T$. For first order hyperbolic problems Ω may as well be an open region. We assume that the initial condition at $t = 0$ is $u(x,y,0) = u_0(x,y)$. The equations (3.1a)-(3.1d) represent a hyperbolic system, if for all real α,β with $\alpha^2 + \beta^2 = 1$, there exists a non-singular transformation matrix P such that $P(\alpha A+\beta B)P^{-1} = D$ where D is a diagonal matrix with real elements. If the matrices A and B in (3.1a)-(3.1d) are symmetric this is sufficient to guarantee that the equations are systems of hyperbolic type, but it is not a necessary condition.

Let $k = 2$ in (1.2), i.e.,

$$f(t,y) = f_1(t,y) + f_2(t,y) \tag{3.2}$$

and consider the 2-stage formula [16]

$$y_{n+1}^{(1)} = y_n + \tau_n[(\lambda-\tfrac{1}{2})f_1(t_n+\alpha\tau_n,y_n) + \tfrac{1}{2}f_1(t_n+\beta\tau_n,y_{n+1}^{(1)}) +$$

(3.3)
$$f_2(t_n+\gamma\tau_n,y_n)],$$

$$y_{n+1} = y_n + \tau_n[(\frac{2\lambda-1}{2\lambda})f_1(t_n+\delta\tau_n,y_n) + \frac{1}{2\lambda}f_1(t_n+\beta\tau_n,y_{n+1}^{(1)}) +$$

$$\tfrac{1}{2}f_2(t_n+\gamma\tau_n,y_n) + \tfrac{1}{2}f_2(t_n+(1-\gamma)\tau_n,y_{n+1})],$$

where the parameters δ and β must satisfy $(2\delta-1)\lambda+\beta-\delta = 0$ for a given λ. The second order consistent formula (3.3) generates several well-known splitting methods.

Replacing the region Ω by a rectangular grid with grid lines parallel to the coordinate axes and semi-discretizing the equations (3.1) using standard central finite differences a system of ODE's (1.1) is obtained. For scalar equations (3.1) the right-hand side function of (1.1) $f(t,y)$ satisfies the 5-point coupling.

3.1. Two alternating direction implicit methods

Let us define the *differential operator splitting* for the equations (3.1a)-(3.1e)

(3.4)
$$f(t,y) = f_1(t,y) + f_2(t,y),$$

where the functions f_1 and f_2 are obtained after semi-discretizing in (3.1) the term with the x-derivative and the term with the y-derivative, respectively.

By substituting these functions into (3.3) with $\lambda=\beta=\frac{1}{2}$ we obtain the ADI method of Peaceman and Rachford [19,20]:

(3.5)
$$y_{n+1}^{(1)} = y_n + \tfrac{1}{2}\tau_n[f_1(t_n+\tfrac{1}{2}\tau_n,y_{n+1}^{(1)}) + f_2(t_n+\gamma\tau_n,y_n)],$$
$$y_{n+1} = y_{n+1}^{(1)} + \tfrac{1}{2}\tau_n[f_1(t_n+\tfrac{1}{2}\tau_n,y_{n+1}^{(1)}) + f_2(t_n+(1-\gamma)\tau_n,y_{n+1})],$$

where $0 \le \gamma \le 1$. Usually the parameter γ is equal to 0 or $\frac{1}{2}$.

Choosing $\lambda = 1$ and $\alpha = \delta = 1 - \beta$ in (3.3) we obtain the ADI method of Douglas and Rachford [3,19]:

$$
\begin{aligned}
y_{n+1}^{(1)} &= y_n + \tau_n[\tfrac{1}{2} f_1(t_n+(1-\beta)\tau_n, y_n) + \tfrac{1}{2} f_1(t_n+\beta\tau_n, y_{n+1}^{(1)}) \\
&\qquad + f_2(t_n+\gamma\tau_n, y_n)], \\
y_{n+1} &= y_{n+1}^{(1)} + \tfrac{\tau_n}{2}[f_2(t_n+(1-\gamma)\tau_n, y_{n+1}) - f_2(t_n+\gamma\tau_n, y_n)],
\end{aligned}
\tag{3.6}
$$

where $0 \le \beta, \gamma \le 1$. Well-known choices for (β,γ) are $(\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},0)$.

For scalar equations (3.1) and system of equations (3.1) the Jacobian matrices of $f_1(t,y)$ and $f_2(t,y)$ are *tridiagonal matrices* and *tridiagonal block matrices* [9,19], respectively. This makes the methods computationally attractive. For the equations (3.1a), (3.1b) and (3.1d) the ADI methods (3.5) and (3.6) require at each time step the solution of systems of linear algebraic equations [9,10,19]. In [5,13] applications of the ADI methods are given for the shallow water equations, i.e. equations of the form (3.1c). In this case systems of non-linear equations have to be solved. For the equation (3.1e) applications of both ADI methods are mentioned in [1]. Mostly the non-linear equations are solved by using a Newton iteration process [13,16].

3.2. The hopscotch methods

In [8] hopscotch algorithms are considered for the numerical solution of a general n-dimensional parabolic equation. It is also possible to apply the hopscotch technique to (nonlinear) hyperbolic first order systems [12]. For a definition of the *odd-even* and *line hopscotch splitting function* we refer to [15,16].

The *odd-even hopscotch method* can be obtained by substituting the odd-even hopscotch splitting function into (3.3) and putting $\lambda = \frac{1}{2}$. The odd-even hopscotch method is suitable for 5-*point coupled* systems (1.1). Then only *scalar implicit equations* have to be solved.

By substituting the line hopscotch splitting function into (3.3) and putting $\lambda = \frac{1}{2}$ the *line hopscotch method* is easily recognized. This method is suitable for 5-*point coupled* and 9-*point coupled* systems (1.1) [16].

In the line hopscotch method only systems of (non-) linear equations with a *tridiagonal Jacobian matrix* have to be solved. To the explicit Strang type splittings[6,25,26], possessing a 9-point coupling, the line hopscotch method can be applied.

The hopscotch methods are suitable when the coupling between the components of f(t,y) is even fully nonlinear. However, in this case these methods seem to be of limited value because of the occurrence of nonlinear instabilities. In the solution of problems with shocks the hopscotch methods have been used in combination with Lax-Wendroff schemes [11].

3.3 A first order method of Douglas and Rachford

Consider the 2-stage formula

$$y_{n+1}^{(1)} = y_n + \tau_n[f_1(t_n+\alpha_1\tau_n, y_{n+1}^{(1)}) + f_2(t_n+\alpha_2\tau_n, y_n)],$$

(3.7)

$$y_{n+1} = y_{n+1}^{(1)} + \tau_n[f_2(t_n+(1-\alpha_2)\tau_n, y_{n+1}) - f_2(t_n+\alpha_2\tau_n, y_n)],$$

where $0 \le \alpha_i \le 1$, $i = 1,2$. For every α_i this method is *first order* consistent. Usually the parameters α_1 and α_2 are equal to 1 and 0, respectively. This splitting formula in combination with the *differential operator splitting* (3.4) was introduced by Douglas and Rachford [3]. In [10] this scheme was suggested for equation (3.1b).

REMARK 3.1. For fully non-linear hyperbolic equations we can formally define the non-linear Peaceman-Rachford method, the non-linear Douglas-Rachford method [16] and the method of successive corrections [15] based on the trapezoidal rule and backward Euler. However these methods are of dubious value for such equations.

4. MULTI-TERM SPLITTING METHODS

Consider the k-dimensional hyperbolic equations

$$u_t = \sum_{i=1}^{k} A_i(x_1,\ldots,x_k)u_{x_i}, \tag{4.1a}$$

$$u_t = \sum_{i=1}^{k} A_i(x_1,\ldots,x_k,t)u_{x_i}, \tag{4.1b}$$

$$u_t = \sum_{i=1}^{k} A_i(u)u_{x_i}, \tag{4.1c}$$

$$u_t = \sum_{i=1}^{k} \frac{\partial G_i(u)}{\partial x_i} \tag{4.1d}$$

with appropriate initial and boundary conditions, i.e. the equations (4.1a)-(4.1d) constitute a properly posed problem in some sense. In general, the unknown u may be either a scalar or vector function of $x_1,\ldots,x_k,t$ and A_i may be a scalar or matrix for $i = 1,\ldots,k$. Assume that standard central finite differences are applied to obtain the semi-discrete system (1.1)-(1.2) where the functions f_i are obained after semi-discretizing in (4.1) the term with the x_i-derivative, i.e. we use the *differential operator splitting* (see section 3.1 and [15,16]). For *scalar equations* (4.1) the right-hand side function of (1.1) f(t,y) satisfies a (2k+1)-*coupling*.

4.1. A method of Gourlay and Mitchell

Let k = 3, and consider the three stage formula

$$\begin{aligned} y_{n+1}^{(1)} &= y_n + \tfrac{1}{2}\tau_n[f_1(t_n+\alpha\tau_n, y_n) + f_2(t_n+\beta\tau_n, y_{n+1}^{(1)})], \\ y_{n+1}^{(2)} &= y_{n+1}^{(1)} + \tfrac{1}{2}\tau_n[f_3(t_n+\gamma\tau_n, y_{n+1}^{(1)}) + f_3(t_n+(1-\gamma)\tau_n, y_{n+1}^{(2)})], \\ y_{n+1} &= y_{n+1}^{(2)} + \tfrac{1}{2}\tau_n[f_2(t_n+(1-\beta)\tau_n, y_{n+1}^{(2)}) + f_1(t_n+(1-\alpha)\tau_n, y_{n+1})], \end{aligned} \tag{4.2}$$

where $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and $0 \le \gamma \le 1$. This splitting method is *second order* consistent and has been suggested by Gourlay and Mitchell [10] for equation (4.1a).

4.2 The locally one-dimensional method of Yanenko

Consider the k-stage formula

$$y_{n+1}^{(0)} = y_n,$$

$$(4.3)\qquad y_{n+1}^{(j)} = y_{n+1}^{(j-1)} + \tau_n[(1-\theta)f_j(t_n+(1-\beta)\tau_n, y_{n+1}^{(j-1)}) + \theta f_j(t_n+\beta\tau_n, y_{n+1}^{(j)})], \qquad j = 1(1)k,$$

$$y_{n+1} = y_{n+1}^{(k)},$$

where β and θ are still free parameters. For every θ and β the method is *first order* consistent. This locally -one- dimensional method emanates from Yanenko [27]. Choosing $\theta = \beta = 1$ we obtain an LOD method which is frequently used in practice. In [10] this splitting method is suggested for hyperbolic equations of the form (4.1b) in two - and three dimensions. Marchuk [18] describes the LOD method(4.3) also for hyperbolic equations (4.1a)-(4.1c).

REMARK 4.1. In [16] several other multi-term splitting methods are discussed for parabolic equations. As far as we know, these multi-term splitting methods have not been discussed in the literature for hyperbolic equations.

5. STABILITY PROPERTIES

In the method of lines approach stability properties can be investigated by making use of matrix theory. If this is not possible then we investigate the stability of a scalar recurrence relation instead of the amplification matrix of the scheme. In this way unconditional stability properties can be shown for several splitting formulas. Partly these results will be based on a theorem which is quite similar to a theorem due to Kellog [17].

5.1 The test-model and a basic theorem

The first order variational equation of (2.1) is of the form

$$(5.1)\qquad y_{n+1} = R(\tau_n J_1, \ldots, \tau_n J_k) y_n,$$

where J_i denotes the Jacobian $\partial f_i/\partial y$ at $(t,y) = (t_n, y_n)$ and the definition of the *matrix-valued stability function* R is given in [16]. This function R can also be obtained by applying (2.1) to the linear equation

$$\frac{dy}{dt} = Jy, \quad J = \sum_{i=1}^{k} J_i. \tag{5.2}$$

If the matrices J_i commute with each other and have a complete set of eigenvectors, which implies that J_i share the same eigensystem, the expression (5.1) can be reduced to the scalar recurrence relation

$$y_{n+1} = R(z_1,\ldots,z_k)y_n, \tag{5.1)'}$$

where z_i represents an eigenvalue of $\tau_n J_i$. The analysis of this *rational function* R is simpler than the analysis of the *amplification matrix* $R(\tau_n J_1,\ldots,\tau_n J_k)$.

For the derivation of stability criteria one must make assumptions on the matrices J and J_i. These are:

(5.3)
1. The matrices J_i are skew-symmetric.
2. The matrices J_i commute.

To interpret these restrictions, consider the k-dimensional equation

$$u_t = \sum_{i=1}^{k} u_{x_i} + \phi(x_1,\ldots,x_k,t) \tag{5.4}$$

on the unit cube with time-dependent Dirichlet boundary conditions. If we impose a uniform grid and semi-discretize using standard finite differences, we obtain a linear system

$$\frac{dy}{dt} = Jy + b(t). \tag{5.5}$$

If we further assume the differential operator splitting, i.e. the matrices J_i are associated to the operators $\partial/\partial x_i$, then the properties (5.3) hold.

The mathematical theory of finite difference approximations to hyperbolic initial-boundary value problems is far from complete. Usually a finite difference scheme is applied in the interior of the grid and then a special difference scheme is used in a neighbourhood of the boundary. Even if the interior scheme is stable for hyperbolic problems, the *boundary treatment* may introduce *instabilities* that make the composite difference method useless. In [2] various interior and boundary scheme combinations are

discussed. For model problems away from the boundaries a Fourier transformation of the space variables is carried out in the method of von Neumann [9,19,22]. Here, we consider the linear equation (5.2), whose matrices satisfy the conditions (5.3), as a *test-model* for stability. This means that only for the interior of the computational grid stability results are derived, i.e. the boundary conditions are neglected.

Note that the conditions (5.3) imply that J_i for $i = 1,\ldots,k$ share the same eigensystem. A skew-symmetric matrix J_i is a normal matrix, i.e. $J_i^T J_i = -J_i J_i = J_i(-J_i) = J_i J_i^T$ where J_i^T denotes the transposed of J_i. Necessary and sufficient for a matrix to have a complete orthogonal set of eigenvectors is that it be normal (cf. [22, p.70]). Thus, the matrices J_i commute and have a complete set of eigenvectors, i.e. the matrices J_i share the same eigensystem (cf. [15]).

For the derivation of the stability criteria for factorized stability functions R we use the following theorem, which is closely related to the theorem of Kellog [17] (see also th. 5.1, [16]).

THEOREM 5.1. *If* S *is a skew-symmetric real* (nxn)*-matrix and* ρ *is a positive scalar, then*

a) $\rho I + S$ *is non-singular,*

b) $\| (\rho I+S)^{-1} \|_2 \leq \rho^{-1}$,

c) $\| (\rho I-S)(\rho I+S)^{-1} \|_2 = 1$.

PROOF. a) It can be easily verified that a skew-symmetric matrix $S(S^T = -S$, where S^T denotes the transposed of S) has purely imaginary eigenvalues. This means that all the eigenvalues z of $\rho I + S$ lie on the line Re $z = \rho$, $\rho > 0$.

b) Let $\rho + i\lambda_s (s=1,\ldots,n, \lambda_s \in \mathbb{R})$ denote the eigenvalues of $\rho I + S$,

then $|\rho+i\lambda_s| \geq \rho \Rightarrow \dfrac{1}{|\rho+i\lambda_s|} \leq \dfrac{1}{\rho}$.

Thus, the absolute value of each eigenvalue of $(\rho I+S)^{-1}$ is less than or equal to $\frac{1}{\rho}$. Further, $(\rho I+S)^{-1}$ is a normal matrix. The spectralnorm of a normal matrix is equal to its spectral radius.

Thus

$$\| (\rho I+S)^{-1} \|_2 \le \max_{1\le s\le n} \frac{1}{|\rho+i\lambda_s|} = \max_{1\le s\le n} \frac{1}{(\rho^2+\lambda_s^2)^{\frac{1}{2}}} = \frac{1}{(\rho^2+\lambda_{s_{min}}^2)^{\frac{1}{2}}},$$

where $|\lambda_{s_{min}}| = \min_{1\le s\le n} |\lambda_s|$. If $\lambda_{s_{min}} = 0$ then $\| (\rho I+S)^{-1} \|_2 \le \frac{1}{\rho}$.

c) Let $M = (\rho I-S)(\rho I+S)^{-1}$ then

$$M^T M = [(\rho I-S)(\rho I+S)^{-1}]^T [\rho I-S][\rho I+S]^{-1} =$$

$$= (\rho I-S)^{-1}(\rho I+S)(\rho I-S)(\rho I+S)^{-1} =$$

$$= (\rho I-S)^{-1}(\rho I-S)(\rho I+S)(\rho I+S)^{-1} = I,$$

i.e. $M^{-1} = M^T$ and M is an orthogonal matrix.
Thus

$$\| M \|_2 = \| (\rho I-S)(\rho I+S)^{-1} \|_2 = 1. \quad \square$$

5.2 Stability Theorems

For the splitting formulas in the sections 3 and 4 stability theorems are listed, which deal with unconditional stability. The results are stated for the *splitting formulas*. Therefore, it is necessary to consider the type of splitting and the underlying class of PDE's before interpreting a result for a splitting method (see section 5.1).

THEOREM 5.2.

1. *The stability function of the formula* (3.3) *reads*

$$R(Z_1,Z_2) = (I-\tfrac{1}{2}Z_2)^{-1}(I-\tfrac{1}{2}Z_1)^{-1}(I+\tfrac{1}{2}Z_1)(I+\tfrac{1}{2}Z_2), \tag{5.6}$$

where $Z_i = \tau_n J_i$, $i = 1,2$.

2. *Let conditions* (5.3) *be satisfied and let* $k = 2$, *then*

$$\| R(\tau_n J_1, \tau_n J_2) \|_2 = 1 \quad \textit{for all } \tau_n > 0.$$

3. *Let* $\tau_n = \tau$, τ *constant. If* J_i, $i = 1,2$, *is skew-symmetric,* $R^n(\tau J_1, \tau J_2)$ *is uniformly bounded in* n *for all* $\tau > 0$.

PROOF. The derivation of the stability function (5.6) is straightforward. Part 2. is easily proved by making use of the commutativity and by observing that the amplification matrix $R(\tau_n J_1, \tau_n J_2)$ is orthogonal (i.e., $R^T(\tau_n J_1, \tau_n J_2)$ $R(\tau_n J_1, \tau_n J_2) = I$). To prove the last result we write $R(\tau J_1, \tau J_2)$ as

$$R(\tau J_1, \tau J_2) = (I - \tfrac{1}{2}\tau J_2)^{-1} \tilde{R}(\tau J_1, \tau J_2)(I - \tfrac{1}{2}\tau J_2),$$

with

$$\tilde{R}(\tau J_1, \tau J_2) = (I - \tfrac{1}{2}\tau J_1)^{-1}(I + \tfrac{1}{2}\tau J_1)(I - \tfrac{1}{2}\tau J_2)^{-1}(I + \tfrac{1}{2}\tau J_2).$$

From part c) of theorem 5.1 it follows that

$$\| \tilde{R}(\tau J_1, \tau J_2) \|_2 = 1.$$

The uniform boundedness of $R^n(\tau J_1, \tau J_2)$ is now obtained from the relation

$$R^n(\tau J_1, \tau J_2) = (I - \tfrac{1}{2}\tau J_2)^{-1} \tilde{R}^n(\tau J_1, \tau J_2)(I - \tfrac{1}{2}\tau J_2)$$

and by part b) of theorem 5.1. □

From theorem 5.2 it follows that the ADI methods of Peaceman-Rachford and Douglas-Rachford, discussed in section 3.1, are unconditionally weakly stable for the equation (5.2) with k = 2 and property (5.3). Although the hopscotch methods, discussed in section 3.2, are based on the same splitting formula as the method of Peaceman-Rachford, the results of theorem 5.2 can not be interpreted in a direct manner for hopscotch splittings (cf. [16]).

THEOREM 5.3.

1. *The stability function of the splitting formula for the alternating direction - locally one-dimensional method* (4.2) *is*

$$(5.7)\qquad R(Z_1,Z_2,Z_3) = (I-\tfrac{1}{2}Z_1)^{-1}(I+\tfrac{1}{2}Z_2)(I-\tfrac{1}{2}Z_3)^{-1}$$
$$(I+\tfrac{1}{2}Z_3)(I-\tfrac{1}{2}Z_2)^{-1}(I+\tfrac{1}{2}Z_1),$$

where $Z_i = \tau_n J_i$ *for* $i = 1,2,3$.

2. *Assume that conditions* (5.3) *with* $k = 3$ *hold, then*

$$||R(\tau_n J_1,\tau_n J_2,\tau_n J_3)||_2 = 1 \text{ for all } \tau_n > 0.$$

PROOF. Part 1. and 2. can be proved in a similar way as in theorem 5.2. □

From theorem 5.2 it follows that the method of Gourlay and Mitchell (4.2) is unconditionally weakly stable for the equation (5.2) with $k = 3$ and property (5.3).

THEOREM 5.4.

1. *The stability function of the splitting formula for the locally one-dimensional method* (4.3) *is*

$$(5.8)\qquad R(Z_1,\ldots,Z_k) = \prod_{i=k}^{1} (I-\theta Z_i)^{-1}(I+(1-\theta)Z_i),$$

where $Z_i = \tau_n J_i$ *for* $i = 1,\ldots,k$.

2. *Let* $\theta = \frac{1}{2}$. *Assume the matrices* J_i *to be skew-symmetric, then*

$$\| R(\tau_n J_1,\ldots,\tau_n J_k)\|_2 = 1 \text{ for all } \tau_n > 0.$$

3. *Let* $\theta = 1$. *Assume the matrices* J_i *to be skew-symmetric, then*

$$\| R(\tau_n J_1,\ldots,\tau_n J_k)\|_2 \le 1 \text{ for all } \tau_n > 0.$$

4. *Let* $\theta = 1$. *Assume the skew-symmetric* (N×N) - *matrices* J_i (*with* N *even*) *are non-singular, then*

$$\|R(\tau_n J_1,\ldots,\tau_n J_k)\|_2 < 1 \text{ for all } \tau_n > 0.$$

PROOF. Part 1. follows again from a trivial calculation. Part 2. is an immediate consequence of part c) of theorem 5.1. Part 3. follows from part b) of theorem 5.1. In part 4. the skew-symmetric (N×N)-matrices J_ℓ are non-singular, i.e.

$$J_\ell x_j = i\lambda_{\ell j} x_j, \quad \ell = 1(1)k, \; j = 1(1)N, \quad x_j \in \mathbb{R}^N, \; \lambda_{\ell j} \in \mathbb{R}\setminus\{0\}.$$

It should be noted that the determinant of a skew-symmetric (N×N)-matrix of odd order (i.e. N is odd) is zero, i.e. the matrix is singular. The amplification matrix $R(\tau_n J_1,\ldots,\tau_n J_k)$ given by (5.8) with $\theta = 1$ is a normal matrix. The spectral norm of a normal matrix is equal to its spectral radius. Thus

$$\|R(\tau_n J_1,\ldots,\tau_n J_k)\|_2 = \max_{1\le j\le N} \left\{ \frac{1}{\prod_{\ell=1}^{k} |1-i\lambda_{\ell j}|} \right\} =$$

$$= \max_{1\le j\le N} \left\{ \frac{1}{\prod_{\ell=1}^{k} (1+\lambda_{\ell j}^2)^{\frac{1}{2}}} \right\} < 1$$

for $\lambda_{\ell j} \neq 0$. □

We see that the locally one-dimensional method is stable under less restrictive conditions than (5.3) (i.e., no commutativity is required).

THEOREM 5.5.

1. *The stability function of the splitting formula for the first order method of Douglas and Rachford* (3.7) *is*

$$R(Z_1,Z_2) = (I-Z_2)^{-1}(I-Z_1)^{-1}(I+Z_2Z_1), \tag{5.9}$$

where $Z_i = \tau_n J_i$ *for* i = 1,2.

2. *Let the matrices* J_j *in* (5.2) *satisfy condition* (5.3) *and let* k = 2, *then the rational stability function* (cf. (5.1)') *is*

$$R = \frac{1-\tau_n^2\lambda_1\lambda_2}{(1-i\tau_n\lambda_1)(1-i\tau_n\lambda_2)}, \tag{5.10}$$

where $i\tau_n\lambda_j$ *represents an eigenvalue of* $\tau_n J_j$, $j = 1,2$, $\lambda_j \in \mathbb{R}$. Further, $|R| \leq 1$ *and if* J_j *for* $j = 1,2$ *is non-singular, then* $|R| < 1$.

PROOF. The derivation of the stability function (5.9) and the rational stability function (5.10) is straightforward. Further,

$$|R|^2 = \frac{1-2\tau_n^2\lambda_1\lambda_2+\tau_n^4\lambda_1^2\lambda_2^2}{1+\tau_n^2(\lambda_1^2+\lambda_2^2)+\tau_n^4\lambda_1^2\lambda_2^2}$$

then using the property $(\lambda_1+\lambda_2)^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 \geq 0$ it follows that $|R| \leq 1$ for $\lambda_1,\lambda_2 \in \mathbb{R}$. If J_j is non-singular for $j = 1,2$, i.e. $(\lambda_1+\lambda_2)^2 > 0$, then $|R| < 1$. □

6. CONCLUDING REMARKS

Each splitting method consists of two components, viz. the splitting function which largely depends on the class of problems under consideration and the splitting formula which can be selected on the ground of accuracy and stability considerations. The most important part of this report is the presentation and formulation of the existing splitting methods. The second order splitting methods in the sections 3 and 4 lead to non-dissipative schemes.

The boundary conditions are not taken into consideration in the stability analysis. Discretizing the boundary conditions in hyperbolic problems is a very difficult part of the semi-discretization process.

Usually in hyperbolic PDE's the boundary conditions are time-dependent. It is well known that splitting methods will usually lose accuracy when the boundary conditions are time-dependent (cf. [4]). This phenomenon was investigated in [4,24] for a class of splitting methods for parabolic PDE's. Following the approach of Sommeijer et al. [24] the boundary-value correction could also be derived for splitting methods for a class of hyperbolic initial-boundary value problems.

The implicit splitting methods discussed in the preceding sections are of dubious value for (fully) nonlinear hyperbolic systems in which discontinuities occur.

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